Hajós' Coloring Conjecture

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Overview

1. Four Color Theorem
2. Hajós Coloring Conjecture
   - Survey
   - Progress on Hajós' Conjecture
   - Future work on Hajós' Conjecture

Four Color Theorem

Definition

A graph is planar if it has a plane embedding (a plane drawing without edge crossing).

Euler's formula for planar graphs

$|F| + |V| = |E| + 2$

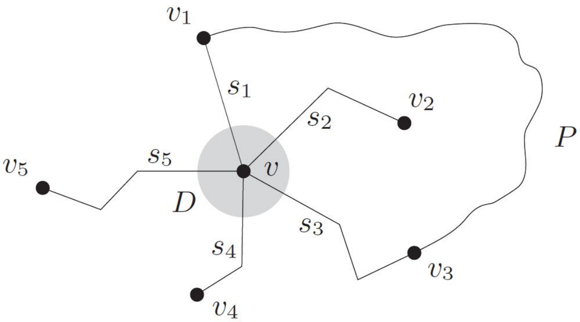
Corollary

- $|E| \leq 3|V| - 6$
- $\delta(G) \leq 5$

Theorem

Every planar graph is 5-colorable.

Four Color Theorem



Four Color Theorem

Four Color Theorem (Appel & Haken, 1977)

Every planar graph is 4-colorable.

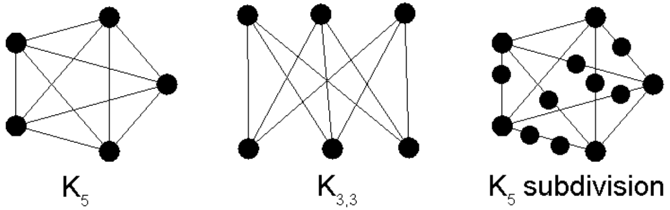
Theorem (Kuratowski, 1930)

Let G be a graph. TFAE

- G is planar.
- G contains no K_5-minor or $K_{3,3}$-minor.
- G contains no K_5-subdivision or $K_{3,3}$-subdivision.

Note: $H-$ subdivision $\subset H-$ minor.

Four Color Theorem



Remark

$\chi(K_{3,3}) = 2$, $\chi(K_5) = 5$.

Question

What's the upper bound of the chromatic number of graphs with no K_5-minor/subdivision?

Hadwiger's Conjecture

Conjecture (Hadwiger, 1943)

For any positive integer k, every graph containing no K_{k+1}-minor is k-colorable.

Survey paper by Paul Seymour:
https://web.math.princeton.edu/~pds/papers/hadwiger/paper.pdf

Survey talk by Zixia Song:
https://www.bilibili.com/video/BV1Ho4y1Z7f7

Hajós' Conjecture - Survey

- Characterization of nonplanar graphs with no $K_{3,3}$-subdivision
- The chromatic number of graphs with no $K_{3,3}$-subdivision is at most 5

Conjecture (Hajós, 1961)

For any positive integer k, every graph containing no K_{k+1}-subdivision is k-colorable.

Counterexamples (Catlin, 1979)

Hajós' conjecture fails for $k \geq 6$.

Theorem (Erdős & Fajtlowicz, 1981)

Hajós' conjecture fails for almost all graphs.

Hajós' Conjecture - Hajós' Graphs

- The conjecture is true for $k \leq 3$.
- Remains open for $k = 4$ and $k = 5$.
- Goal: solve the conjecture for $k = 4$!!!

Definition

We say that a graph G is a *Hajós graph* if

(1) G contains no K_5-subdivision,

(2) G is not 4-colorable, i.e., $\chi(G) \geq 5$, and

(3) subject to (1) and (2), $|V(G)|$ is minimum.

Question

How to reduce the size of G?

Operations

(1) Delete a few vertices
(2) Contract edges
(3) Identify 2 vertices that are not adjacent
(4) Separate G into several pieces

Hajós' Conjecture - Connectivity

Kelmans-Seymour conjecture / Theorem (He, Wang & Yu, 2018+)

Every 5-connected nonplanar graph contains a K_5-subdivision.

Hajós' Graph is not 5-connected (by Kelmans-Seymour conjecture and the Four Color Theorem)

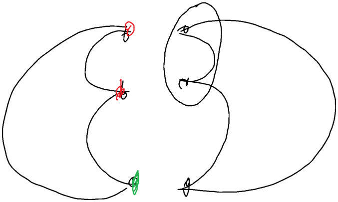
Theorem (Yu & Zickfeld, 2006)

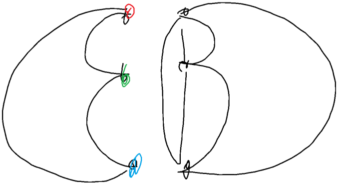
Hajós' Graph must be 4-connected.

Theorem (Sun & Yu, 2016)

Let S be a 4-cut of a Hajós' Graph G. Then $G - S$ has exactly 2 components.

Hajós' Conjecture - Connectivity





Hajós' Conjecture - Main Results

Definition (Separation)

A separation in a graph G consists of a pair of subgraphs G_1, G_2, denoted as (G_1, G_2), such that $G = G_1 \cup G_2$, $E(G_1 \cap G_2) = \emptyset$, and, for $i = 1, 2$, $V(G_i) - V(G_{3-i}) \neq \emptyset$ or $E(G_i) \neq \emptyset$. The order of the separation is $|V(G1 \cap G2)|$, and (G_1, G_2) is said to be a k-separation if its order is k.

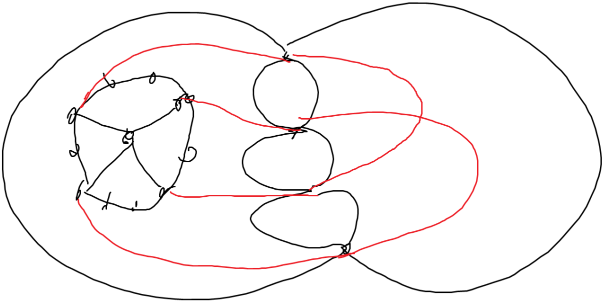
Definition ((G, S)-planar)

Let $S \subseteq V(G)$. A disc representation of a graph G is a drawing of G in a closed disc without edge-crossing. We say that (G, S) is planar if S are vertices in G such that G has a disc representation with S on the boundary of the disc.

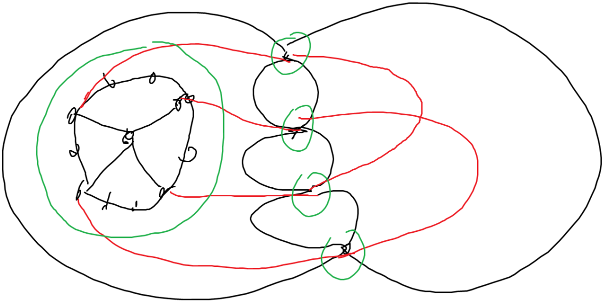
Theorem (Xie, Xie, Yu & Yuan, 2021+)

No Hajós graph has a 4-separation (G_1, G_2) such that $(G_1, V(G_1 \cap G_2))$ is planar and $|V(G_1)| \geq 6$.

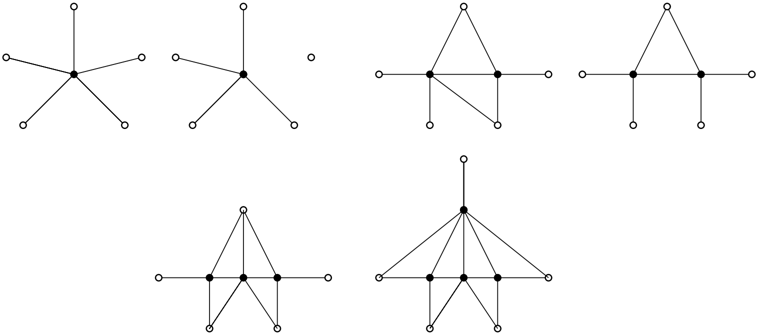
Proof Sketch



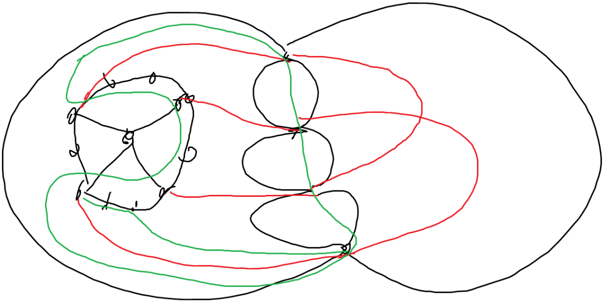
Proof Sketch



Proof Sketch



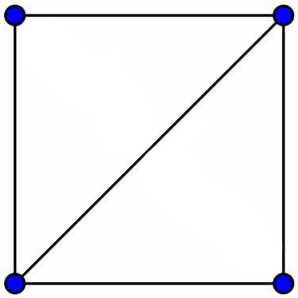
Proof Sketch



Hajós' Conjecture - Future Work

Conjecture

No Hajós graph contains K_4^- as a subgraph.



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